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Generalized Solutions of Wick-type Stochastic KdV-Burgers Equations Using Exp-function Method

Hossam A. Ghany[1,2] and M. Zakarya[3]
[1]Department of Mathematics, Faculty of Science Taif University, Taif, Saudi Arabia.
[2]Department of Mathematics, Helwan University, Cairo, Egypt. h.abdelghany@yahoo.com
[3]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt.
mohammed zakarya@rocketmail.com

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Abstract: *Variable coefficients and Wick-type stochastic KdV-Burgers equations are researched. Exp-function method is proposed to present soliton and periodic wave solutions for variable coefficients KdV-Burgers equation. Generalized white noise functional solutions for Wick-type stochastic KdV-Burgers equations are showed via Hermite transform and white noise analysis.*

Keywords: *KdV-Burgers equation; Exp-function method; Wick product; Hermite transform; White noise.*

Introduction

In this paper, we investigate the variable coefficients KdV-Burgers equation:

$$u_t + f(t)uu_x - g(t)u_{xx} + h(t)u_{xxx} = 0, \qquad (t,x) \in R_+ \times R, \quad (1.1)$$

where $f(t)$, $g(t)$ and $h(t)$ are bounded measurable or integrable functions on R_+. $Eq.(1.1)$ arises in many physical phenomena, such as the propagation of waves in an elastic tube filled with a viscous fluid [25], the flow of liquids containing gas bubbles [29], turbulence [9] and ferroelectricity [32,33]. If such physical phenomena are considered in random environment, we can get random KdV-Burgers equation. In order to give the exact solutions of this random model, we only consider it in white noise environment, that is, we will study the following Wick-type stochastic KdV-Burgers equations:

$$U_t + F(t) \diamond U \diamond U_x - G(t) \diamond U_{xx} + H(t) \diamond U_{xxx} = 0 \qquad (1.2)$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space $(s)_{-1}$ and $F(t)$, $G(t)$ and $H(t)$ are $(s)_{-1}$ -valued functions [24]. It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in random environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. Wadati [28] first answered the interesting question, "How does external noise affect the motion of solitons?"and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. The cauchy problems associated with stochastic partial differential equations $(PDEs)$ was discussed by many authors, e.g., de Bouard and Debussche [4,5], Debussche and Printems [6,7], Printems [27] and Ghany and Hyder [13]. On the basis of white noise functional analysis [24], Ghany et al. [10–12,14–17] studied more intensely the white noise functional solutions for some nonlinear stochastic PDEs. Recently, many new methods have been proposed to solve the non-linear wave equations such as variational iteration method [19, 26], tanh-function method [2, 21], homotopy perturbation method [8, 18] and F-expansion method [1]. The Exp-function method was first proposed in [20]. As it is a straightforward and concise method ,it was successfully applied to obtain generalized solitary solutions and periodic solutions of some nonlinear evolution equation arising in mathematical physics. The application of this method can be found in [22, 30, 35, 36]. Moreover, The solution procedure of this method, with the aid of *Maple*, is of utter simplicity and this method can be easily extended to other kinds of nonlinear evolution equations [23, 34]. In our paper, we use the Exp-function method to seek new exact travelling wave solutions for the variable coefficients KdV-Burgers equation. These solutions include soliton and periodic wave solutions. Then, with the help of Hermit transform and white noise analysis, we employ these solutions to find generalized white noise functional solutions for the Wick-type stochastic KdV-Burgers equations**.**

Soliton and Periodic Wave Solutions of Eq.(1.1)

In this section, we apply Hermite transform, white noise theory, and exp-function method to explore soliton and periodic wave solutions for Eq.(1.1). Applying Hermite transform to Eq.(1.2), we get the deterministic equation:

$$\tilde{U}_t(t,x,z) + \tilde{F}(t,z)\,\tilde{U}(t,x,z)\,\tilde{U}_x(t,x,z) - \tilde{G}(t,z)\,\tilde{U}_{xx}(t,x,z) + \tilde{H}(t,z)\,\tilde{U}_{xxx}(t,x,z) = 0 \qquad (2.1)$$

where $z = (z1, z2, \dots) \in (C^N)$ is a vector parameter. To look for the traveling wave solution of $Eq.(2.1)$, we make the transformations $\tilde{F}(t,z) := f(t,z)$, $\tilde{G}(t,z) := g(t,z)$, $\tilde{H}(t,z) := h(t,z)$ and $\tilde{U}(t,x,z) =: u(t,x,z) = u(\xi(t,x,z))$ with

$$\xi(t,x,z) = kx + \int_0^t \omega(\tau,z)d\tau + c,$$

where $k \neq 0$, c are arbitrary constants and $\omega(\tau,z)$ is a nonzero function of the indicated variables to be determined later. Hence, $Eq.(2.1)$ can be transformed into the following ordinary differential equation:

$$\omega u' + fkuu' - gK^2u'' + hK^3u''' = 0, \qquad (2.2)$$

where the prime denote to the differential with respect to ξ. In view of exp-function method, the solution of $Eq.(2.1)$, can be expressed in the form:

$$u(t,x,z) = \frac{\sum_{n=-c}^{d} an \exp(n\xi)}{\sum_{m=-p}^{q} bn \exp(m\xi)}, \qquad (2.3)$$

where $c,\ d,\ p$ and q are positive integers which could be freely chosen and $an,\ bm$ are unknown constants to be determined later. $Eq.(2.3)$ can be re-written in an alternative form as follows:

$$u(t,x,z) = \frac{ac\exp(c\xi) + \cdots + a_{-d}\exp(-d\xi)}{bp\exp(n\xi) + \cdots + b_{-q}\exp(-q\xi)}, \qquad (2.4)$$

To determine the values of c and p, we balance the linear term of highest order of $Eq.(2.2)$ with the highest order nonlinear term. By simple calculation, we have

$$u''' = \frac{c1\exp[(c+3p)\xi)]+\cdots}{c2\exp[4p\xi]+\cdots}, \qquad (2.5)$$

$$uu' = \frac{c3\exp[(2c+2p)\xi)]+\cdots}{c4\exp[4p\xi]+\cdots}, \qquad (2.6)$$

where ci are determined coefficients only for simplicity. Balancing highest order of exponential function in $Eqs.(2.5),\ (2.6)$, we have $p = c$, similarly to determine values of d and q, we balance the linear term of lowest order in $Eq.(2.2)$:

$$u''' = \frac{\ldots + d1\exp[-(q+3d)\xi)]}{\ldots + d2\exp[-4q\xi]}, \qquad (2.7)$$

$$uu' = \frac{\ldots + d3\exp[-(2q+2d)\xi)]}{\ldots + d4\exp[-4q\xi]}, \qquad (2.8)$$

where di are determined coefficients only for simplicity. Therefore we can obtain $d = q$. Now, we solve $Eq.(2.1)$ for some particular cases for the constants $p,\ c, d\ and\ q$.

Case A.

If we set $p = c = 1$ and $d = q = 1$, then $Eq.(2.4)$ becomes

$$u(t,x,z) = \frac{\begin{bmatrix} a1\exp(\xi(t,x,z)) + a0 \\ + a_{-1}\exp(-\xi(t,x,z)) \end{bmatrix}}{\begin{bmatrix} b1\exp(\xi(t,x,z)) + b0 \\ + b_{-1}\exp(-\xi(t,x,z)) \end{bmatrix}}, \qquad (2.9)$$

Substituting $Eq.(2.9)$ into (2.2), using *Maple*, and equating to zero the coefficients of all powers of $exp(n\xi)$, yield a system of algebraic equations in the unknowns $a0,\ b0,\ a1,\ a-1,\ b-1$ and ω in the form.

$$-\omega a_0 - fk^3 a_0 - gk^2 a_0 + hk^3 a_1 b_0 + 9k^2 a_1 b_0 + fka_1^2 b_0 - fka_1 b_0 + \omega a_1 b_0 = 0\,,$$

$$-4gk^2 a_{-1} - 8hk^3 a_{-1} - fka_0^2 + 4hk^3 a_0 b_0 - 2fka_1 a_{-1} + 2fka_1^2 b_{-1} + fka_1^2 b_0^2 - 2\omega a_0 b_0 + 2\omega a_1 b_0^2 + 2\omega a_1 b_{-1} - 4k^3 h a_1 b_0^2 + 4gk^2 a_1 b_{-1} + 8hk^3 a_1 b_{-1} - 2\omega a_{-1} = 0,$$

$$-hk^3 a_0 b_0^2 - 18hk^3 a_1 b_{-1} b_0 - 5hk^3 a_{-1} b_0 + hk^3 a_1 b_0^3 - \omega a_0 b_{-1} + 2gk^2 a_1 b_{-1} b_0 + \omega a_1 b_0^3 + 23hk^3 a_0 b_{-1} - 5\omega a_{-1} b_0 + 2fka_0 a_1 b_{-1} + 3fk\, a_1^2 b_{-1} b_0 - 2fka_1 a_{-1} b_0 + fka_0 a_1 b_0^2 + hk^2$$

$$a_0b_0^2 - 7gk^2a_{-1}b_0 + 5gk^2a_0b_{-1} - g$$
$$k^2a_1b_0^3 + 6\omega a_1b_{-1}b_0 - fka_0^2b_0 - 3f$$
$$ka_0a_{-1} - \omega a_0b_0^2 = 0,$$

$$-\omega a_0b_{-1}^2 - hk^3a_{-1}b_0^3 + gk^2a_0b_{-1}b_0^2$$
$$-23hk^3a_0b_{-1}^2 + 2fka_{-1}a_1b_{-1}b_0 + f$$
$$ka_0^2b_{-1}b_0 + \omega a_0b_{-1}b_0^2 + 5\omega a_1b_{-1}^2b_0$$
$$+18hk^3a_{-1}b_{-1}b_0 - 6\omega a_{-1}b_{-1}b_0 + h$$
$$k^3a_0b_{-1}b_0^2 - 3fka_{-1}^2b_0 - \omega a_{-1}b_0^3 -$$
$$7gk^2a_1b_{-1}^2b_0 + 2gk^2a_{-1}b_{-1}b_0 + 5h$$
$$k^3a_1b_{-1}^2b_0 - gk^2a_{-1}b_0^3 + 5gk^2a_0b_0^2$$
$$+3fka_0b_{-1}^2a_{-1} + 2fka_0b_{-1}a_{-1} - fk$$
$$a_0b_{-1}^2a_{-1} = 0,$$

$$-2\omega b_1^2a_{-1} - 4hk^3a_0b_{-1}^2b_0 + 2\omega a_1$$
$$b_1^3 - 2\omega a_{-1}b_{-1}b_0^2 + fka_0^2b_1^2 + 2fk$$
$$a_{-1}a_1b_1^2 + 4gk^3a_{-1}b_1^2 + 4hk^3a_{-1}$$
$$b_{-1}b_0^2 + 8hk^3a_1a_{-1}^3 + 2\omega a_0b_{-1}^2b_0$$
$$-8fk^3a_{-1}b_{-1}^2 - 4gk^2a_1b_{-1}^3 - 2fk$$
$$a_{-1}^2b_{-1} \quad -fka_{-1}^2b_0^2 = 0,$$

$$\omega a_0b_{-1}^3 - gk^2a_0b_{-1}^3 - \omega a_{-1}b_{-1}^2b_0 +$$
$$+gk^2a_{-1}b_{-1}^2b_0 - fka_{-1}^2b_{-1}b_0 + fk$$
$$a_{-1}a_0b_{-1}^2 - hk^3a_{-1}b_{-1}^2b_0 + hk^3a_0$$
$$b_{-1}^3 = 0\,.$$

(2.10)

Solving this system with aid of *Mable*, we get the coefficients:

$$\begin{cases} a_0 = \dfrac{4a_{-1}}{b_0},\quad b_{-1} = \dfrac{b_0^2}{4}, \\ a_1 = \dfrac{4}{25}\,\dfrac{\begin{bmatrix}25\,a_{-1}f(t,z)h(t,z)\\ -3b_0^2g^2(t,z)\end{bmatrix}}{b_0^2f(t,z)}, \\ \omega = -\dfrac{2k}{25}\,\dfrac{\begin{bmatrix}50\,a_{-1}f(t,z)h(t,z)\\ -3b_0^2g^2(t,z)\end{bmatrix}}{b_0^2f(t,z)} \end{cases} \qquad (2.11)$$

where $a-1$ and $b0$ are free parameter-s. Substituting these values into $Eq.(2.9)$, we obtain the following soliton wave solution of $Eq.(2.1)$.

$$u1(t,x,z) = \frac{\left[\nabla_1 + \frac{4a_{-1}}{b_0} + a_{-1}\exp(-\xi_1(t,x,z))\right]}{\left[\exp(\xi_1(t,x,z)) + b_0 + \frac{b_0^2}{4}\exp(-\xi_1(t,x,z))\right]}, \qquad (2.12)$$

Where

$$\nabla_1 = \frac{4}{25}\left[\frac{\left[25\, a_{-1} f(t,z)h(t,z) - 3b_0^2 g^2(t,z)\right]}{b_0^2 f(t,z)}\right] \exp(\xi_1(t,x,z)),$$

$$\xi_1(t,x,z) = k\left[x - \frac{2}{25}\ \nabla_2\right], \qquad (2.13)$$

$$\nabla_2 = \int_0^t \left[\frac{\left[50\, a_{-1} f(t,z)h(t,z) - 3b_0^2 g^2(t,z)\right]}{b_0^2 f(t,z)}\right] d\tau$$

In the case of k is an imaginary number, i.e., $k = iK$, the above soliton wave solution can be converted into a periodic wave solution of $Eq. (2.1)$ as follows:

$$u_2(t,x,z) = \frac{\left[\nabla_3 \cos(\xi_1(t,x,z)) + \frac{4a_{-1}}{b_0} + i\ \nabla_4 \sin(\xi_1(t,x,z))\right]}{\left[\left[1 + \frac{b_0^2}{4}\right]\cos(\xi_1(t,x,z)) \quad \left[1 - \frac{b_0^2}{4}\right] i \sin(\xi_1(t,x,z))\right]}, (2.14)$$

Where

$$\nabla_3 = \left[\frac{4}{25}\left[\frac{\left[25\, a_{-1} f(t,z)h(t,z) - 3b_0^2 g^2(t,z)\right]}{b_0^2 f(t,z)}\right] + a_{-1}\right],$$

$$\nabla_4 = \left[\frac{4}{25}\left[\frac{\left[25\, a_{-1} f(t,z)h(t,z) - 3b_0^2 g^2(t,z)\right]}{b_0^2 f(t,z)}\right] - a_{-1}\right].$$

where $\xi_1(t,x,z)$ is expressed by $Eq. (2.13)$. If we search for a periodic wave solution, the imaginary part in the denominator or of $Eq. (2.14)$ must be $zero$, that requires that $b_0 = \pm 2$. so, we have a periodic wave solution of $Eq. (2.1)$ as the form:

$$u_2^{*} = \frac{\nabla_5 \cos\left(\xi_1^{*}(t,x,z)\right) \pm 2a_{-1}}{2\cos\left(\xi_1^{*}(t,x,z)\right) \pm 2a_{-1} \pm 2}, \quad (2.15)$$

Where

$$\nabla_5 = \left[\frac{1}{25}\left[\frac{\begin{bmatrix}25\, a_{-1} f(t,z) h(t,z)\\ -12 g^2(t,z)\end{bmatrix}}{f(t,z)}\right] + a_{-1}\right],$$

$$\xi_1^{*}(t,x,z) = k\left[x - \frac{1}{25}\ \nabla_6\right], \quad (2.16)$$

$$\nabla_6 = \int_0^t \left[\frac{\begin{matrix}25\, a_{-1} f(t,z) h(t,z)\\ -6 g^2(t,z)\end{matrix}}{f(t,z)}\right] d\tau,$$

Case B.

If we set $p = c = 2$ and $d = q = 1$, then $Eq. (2.4)$ becomes

$$u(t,x,z) = \frac{\begin{bmatrix} a_{-2}\exp\left(-2\xi(t,x,z)\right) \\ a_1 \exp\left(\xi(t,x,z)\right) + a0 \\ + a_{-1}\exp\left(-\xi(t,x,z)\right)\end{bmatrix}}{\begin{bmatrix}\exp\left(-2\xi(t,x,z)\right) + \\ b_1 \exp\left(\xi(t,x,z)\right) + b0 \\ + b_{-1}\exp\left(-\xi(t,x,z)\right)\end{bmatrix}}, \quad (2.17)$$

Substituting $\text{Eq. } (2.17)$ in $\text{Eq. } (2.2)$, using *Maple*, and equating to zero the coefficients of all powers of $exp(n\xi)$, yield a system of algebraic equations in the unknowns $a0,\ b0,\ a1,\ b1,\ b-1,\ a-1,\ a-2$ and ω. solving this system with aid of *Maple*, we get the coefficients:

$$\begin{cases} b_{-1} = \dfrac{-4b_1^3}{27}, a0 = b0 = 0, \\ a_1 = \dfrac{\begin{bmatrix}16 b_1^3 g^2(t,z) - 675 \\ a_{-1} f(t,z) h(t,z)\end{bmatrix}}{100\, b_1^2 f(t,z) h(t,z)}, \\ a_{-2} = -\dfrac{\begin{bmatrix}48\, b_1^3 g^2(t,z) + 675 \\ a_{-1} f(t,z) h(t,z)\end{bmatrix}}{100\, b_1^3 f(t,z) h(t,z)}, \\ \omega = 3k\left[\dfrac{\begin{bmatrix}8\, b_1^3 g^2(t,z) + 225 \\ a_{-1} f(t,z) h(t,z)\end{bmatrix}}{100\, b_1^3 h(t,z)}\right]. \end{cases} \quad (2.18)$$

where a_{-1} and b_1 are free parameters. Substituting these values into $Eq.(2.17)$, we obtain the following soliton wave solution of $Eq.(2.1)$

$$u_3(t,x,z) = \frac{\left[\begin{array}{c}\nabla_7 + \nabla_8 + a_{-1} \\ \exp\left(-\xi_2(t,x,z)\right)\end{array}\right]}{\left[\begin{array}{c}\exp\left(-2\xi_2(t,x,z)\right) + \\ b_1 \exp\left(\xi_2(t,x,z)\right) - \\ \frac{4b_1^3}{27}\exp\left(-\xi_2(t,x,z)\right)\end{array}\right]}, \quad (2.19)$$

Where

$$\nabla_7 = -\frac{\left[\begin{array}{c}48\, b_1^3 g^2(t,z) + 675 \\ a_{-1} f(t,z)\, h(t,z)\end{array}\right]}{100\, b_1^3 f(t,z) h(t,z)} \exp(-2\xi_2),$$

$$\nabla_8 = \frac{\left[\begin{array}{c}16\, b_1^3 g^2(t,z) - 675 \\ a_{-1} f(t,z) h(t,z)\end{array}\right]}{100\, b_1^2 f(t,z) h(t,z)} \exp(\xi_2),$$

$$\xi_2(t,x,z) = k\left[x + \frac{3}{100}\ \nabla_9\right], \quad (2.20)$$

$$\nabla_9 = \int_0^t \left[\frac{\left[\begin{array}{c}8\, b_1^3 g^2(t,z) + 225 \\ a_{-1} f(t,z) h(t,z)\end{array}\right]}{b_1^3 h(t,z)}\right] d\tau.$$

According to Case A, $Eq.(2.19)$ can be converted into a periodic wave solution as follows:

$$u_4(t,x,z) = \frac{\left[\begin{array}{c}\nabla_{10} + \nabla_{11}\cos\left(\xi_2(t,x,z)\right) \\ +\nabla_{12}\ \mathrm{i}\sin\left(\xi_2(t,x,z)\right)\end{array}\right]}{\left[\Delta_1 + b_1\Delta_2 - \frac{4b_1^3}{27}\Delta_3\right]}, \quad (2.21)$$

Where

$$\nabla_{10} = -\frac{\left[\begin{array}{c}48\, b_1^3 g^2(t,z) + 675 \\ a_{-1} f(t,z) h(t,z)\end{array}\right]}{100 b_1^3 f(t,z) h(t,z)}\ \Delta_1,$$

$$\nabla_{11} = \left[\frac{\left[\begin{array}{c}16\, b_1^3 g^2(t,z) - 675 \\ a_{-1} f(t,z) h(t,z)\end{array}\right]}{100 b_1^2 f(t,z) h(t,z)} + a_{-1}\right],$$

$$\nabla_{12} = \left[\frac{\left[\begin{array}{c}16\, b_1^3 g^2(t,z) - 675 \\ a_{-1} f(t,z) h(t,z)\end{array}\right]}{100 b_1^2 f(t,z) h(t,z)} - a_{-1}\right]$$

And

$\Delta_1 = \cos(2\xi_2) - i\sin(2\xi_2)$,

$\Delta_2 = \cos(\xi_2) + i\sin(2\xi_2)$,

$\Delta_2 = \cos(\xi_2) - i\sin(2\xi_2)$,

where $\xi_2(t,x,z)$ is expressed by $Eq.(2.20)$.

Case C.

If we set $p = c = 2$ and $d = q = 2$, then $Eq.(2.4)$ becomes.

$$u(t,x,z) = \frac{\begin{bmatrix} a_2\exp\left(2\xi(t,x,z)\right) \\ a_1\exp\left(\xi(t,x,z)\right) + a0 \\ + a_{-1}\exp\left(-\xi(t,x,z)\right) \\ + a_{-2}\exp\left(-2\xi(t,x,z)\right) \end{bmatrix}}{\begin{bmatrix} \exp\left(2\xi(t,x,z)\right) + \\ b_1\exp\left(\xi(t,x,z)\right) + b0 \\ + b_{-1}\exp\left(-\xi(t,x,z)\right) \\ + b_{-2}\exp\left(-2\xi(t,x,z)\right) \end{bmatrix}}, \quad (2.22)$$

Substituting $Eq.(2.22)$ into $Eq.(2.2)$, using Maple, and equating to *zero* the coefficients of all powers of $exp(n\_\xi)$, yield a system of algebraic equations in the unknowns $a0,\ a1,\ a2,\ a_{-1},\ a_{-2},\ b0,\ b1,\ b_{-1},\ b_{-2}$ and ω. solving this system with aid of *Maple*, we get the coefficients.

$$\begin{cases} a_1 = a_{-1} = b0 = b_1 = b_{-1} = 0, \\ a0 = \dfrac{a_{-2}b0}{b_{-2}}, \\ a_2 = \dfrac{-8kb_{-2}\,g(t,z) + a_{-1}f(t,z)}{b_{-2}f(t,z)}, \\ \omega = -k\left[\dfrac{\begin{bmatrix} -4kb_{-2}\,g(t,z) + a_{-1} \\ f(t,z) + 16k^2b_{-2}h(t,z) \end{bmatrix}}{b_{-2}}\right] \end{cases}$$

where a_{-2} and b_{-2} are free parameters. Substituting these values into $Eq.(2.22)$, we obtain the following soliton wave solution of $Eq.(2.1)$

$$u_5(t,x,z) = \frac{\left[V_{13} + a_{-2}\exp\left(-2\xi_3(t,x,z)\right)\right]}{\begin{bmatrix} \exp\left(2\xi_3(t,x,z)\right) \\ + b_{-2}\exp\left(-2\xi_3(t,x,z)\right) \end{bmatrix}}, (2.24)$$

Where

$$\nabla_{13} = \frac{\begin{bmatrix} -8k\,b_{-2}\, g(t,z) \\ +a_{-1} f(t,z) \end{bmatrix}}{b_{-2} f(t,z)} \exp(2\xi_3(t,x,z)),$$

$$\xi_3(t,x,z) = k[x - \nabla_{14}], \qquad (2.25)$$

$$\nabla_{14} = \int_0^t \left[\frac{\begin{bmatrix} -4k b_{-2}\, g(t,z) + a_{-1} \\ f(t,z) + 16k^2 b_{-2} h(t,z) \end{bmatrix}}{b_{-2}} \right] d\tau.$$

According to Case A, $Eq.(2.24)$ can be converted into a periodic wave solution as follows:

$$u_6(t,x,z) = \frac{\begin{bmatrix} \nabla_{15} \cos(2\xi_3(t,x,z)) \\ +i\, \nabla_{16} \sin(2\xi_3(t,x,z)) \end{bmatrix}}{\begin{bmatrix} [1 + b_{-2}]\cos(2\xi_3(t,x,z)) \\ +i[1 - b_{-2}]\sin(2\xi_3(t,x,z)) \end{bmatrix}}, (2.26)$$

$$\nabla_{15} = \left[\frac{\begin{bmatrix} -8k b_{-2}\, g(t,z) \\ +a_{-1} f(t,z) \end{bmatrix}}{b_{-2} f(t,z)} + a_{-2} \right],$$

$$\nabla_{16} = \left[\frac{\begin{bmatrix} -8k b_{-2}\, g(t,z) \\ +a_{-1} f(t,z) \end{bmatrix}}{b_{-2} f(t,z)} - a_{-2} \right],$$

where $\xi_3(t,x,z)$ is expressed by $Eq.(2.25)$.

Obviously, there are infinitely number of soliton and periodic wave solutions for $Eq.(2.1)$. These solutions come from setting different values for the positive integers p, c, d and q. The above mentioned cases are just to clarify how far our technique is applicable.

Generalized White Noise Functional Solutions of Eq. (1.2)

In this section, we employ the results of Section 2 and Hermite transform to obtain generalized white noise functional solutions for Wick-type stochastic KdV-Burgers equations (1.2). The properties of exponential and trigonometric functions yield that there exists a bounded open set $\boldsymbol{D} \subset R_+ \times R$, $\rho < \infty, \delta > 0$ such that the solution $u(t,x,z)$ of $Eq.(2.1)$ and all its partial derivatives which are involved in $Eq.(2.1)$ are uniformly bounded for $(t,x,z) \in \boldsymbol{D} \times K_\rho(\delta)$, continuous with respect to $(t,x) \in \boldsymbol{D}$ for all $z \in K_\rho(\delta)$ and analytic with respect to $z \in K_\rho(\delta)$,for all $(t,x) \in \boldsymbol{D}$. From Theorem 4.1.1 *in* [24], there exists $U(t,x,z) \in (S)_{-1}$ such that $u(t,x,z)\, \tilde{U}(t,x)\,(z)$ for all

$(t,x,z) \in \boldsymbol{D} \times K\rho(\delta)$, and $U(t,x)$ solves $Eq.(1.2)$ in $(S)_{-1}$ Hence, by applying the inverse Hermite transform to the results of Section 2, we get the generalized white noise functional solutions of $Eq.(1.2)$ as follows:

- **Generalized stochastic soliton solutions:**

$$U1(t,x) = \frac{\begin{bmatrix} \nabla_{17} + \frac{4a_{-1}}{b_0} + \\ a_{-1}\exp^{\diamond}(-\Xi_1(t,x)) \end{bmatrix}}{\begin{bmatrix} \exp^{\diamond}((\Xi_1(t,x)) + b_0 \\ + \frac{b_0^2}{4}\exp^{\diamond}(-(\Xi_1(t,x)) \end{bmatrix}}, \quad (3.1)$$

Where

$$\nabla_{17} = \frac{4}{25}\,\frac{\begin{bmatrix} 25\ a_{-1}F(t) \diamond H(t) \\ -3b_0^2 G^{\diamond 2}(t) \end{bmatrix}}{b_0^2 F(t)} \diamond \exp^{\diamond}(\Xi_1(t,x)),$$

$$U_2(t,x) = \frac{\begin{vmatrix} \nabla_{18} + \nabla_{19} + a_{-1} \\ \exp^{\diamond}(-\Xi_2(t,x,)) \end{vmatrix}}{\begin{bmatrix} \exp^{\diamond}(-2\Xi_2(t,x)) + \\ b_1\exp^{\diamond}(\Xi_2(t,x)) - \\ \frac{4b_1^3}{27}\exp^{\diamond}(-\Xi_2(t,x)) \end{bmatrix}}, (3.2)$$

Where

$$\boldsymbol{\nabla}_{18} = -\left[\frac{\begin{bmatrix} \mathbf{48}\ b_1^3 G^{\diamond 2}(t) + \mathbf{675} \\ a_{-1}F(t) \diamond H(t) \end{bmatrix}}{\mathbf{100}b_1^3 F(t) \diamond H(t)}\right] \diamond \mathbf{exp}^{\diamond}(-2\Xi_2(t,x)),$$

$$\boldsymbol{\nabla}_{19} = \left[\frac{\begin{bmatrix} \mathbf{16}\ b_1^3 G^{\diamond 2}(t) - \mathbf{675} \\ a_{-1}F(t) \diamond H(t) \end{bmatrix}}{\mathbf{100}b_1^2 F(t) \diamond H(t)}\right] \diamond \mathbf{exp}^{\diamond}(\Xi_2(t,x)),$$

$$\boldsymbol{U}_3(t,x) = \frac{|\nabla_{20} + a_{-2}\exp^{\diamond}(-2\Xi_3(t,x,))|}{\begin{bmatrix} \exp^{\diamond}(2\Xi_3(t,x)) \\ +\mathbf{b}_{-2}\exp^{\diamond}(-2\Xi_3(t,x,)) \end{bmatrix}}, (\mathbf{3.3})$$

where

$$\boldsymbol{\nabla}_{20} = \frac{\begin{bmatrix} -8\mathbf{b}_{-2}\ G(t) \\ +a_{-1}F(t) \end{bmatrix}}{\mathbf{b}_{-2}F(t)} \diamond \exp^{\diamond}(2\Xi_3(t,x)),$$

• Generalized stochastic periodic solutions:

$$U_4(t,x) = \frac{\left[\frac{4a_{-1}}{b_0} \begin{matrix} \nabla_{21} \diamond \cos^{\diamond}((\Xi_1(t,x)) + \\ +i\ \nabla_{22} \diamond \sin^{\diamond}((\Xi_1(t,x)) \end{matrix}\right]}{\left[\begin{matrix} \left[1+\frac{b_0^2}{4}\right]\cos^{\diamond}((\Xi_1(t,x)) \\ \left[1-\frac{b_0^2}{4}\right] i\sin^{\diamond}((\Xi_1(t,x)) \end{matrix}\right]}, (3.4)$$

Where

$$\nabla_{21} = \left[\frac{4}{25}\left[\frac{\begin{bmatrix}25\ a_{-1}F(t)\diamond H(t) \\ -3b_0^2 G^{\diamond 2}(t)\end{bmatrix}}{b_0^2 F(t)}\right] + a_{-1}\right],$$

$$\nabla_{22} = \left[\frac{4}{25}\left[\frac{\begin{bmatrix}25\ a_{-1}F(t)\diamond H(t) \\ -3b_0^2 G^{\diamond 2}(t)\end{bmatrix}}{b_0^2 F(t)}\right] - a_{-1}\right].$$

$$U_5(t,x) = \frac{\begin{bmatrix}\nabla_{23} + \nabla_{24} \diamond \cos^{\diamond}(\Xi_2(t,x)) \\ +i\ \nabla_{25} \diamond \sin^{\diamond}(\Xi_2(t,x))\end{bmatrix}}{\left[\Delta_4 + b_1\Delta_5 - \frac{4b_1^3}{27}\Delta_6\right]}, \quad (3.5)$$

Where

$$\nabla_{23} = -\frac{\begin{bmatrix}48\ b_1^3 G^{\diamond 2}(t) + 675 \\ a_{-1}F(t) \diamond H(t)\end{bmatrix}}{100\, b_1^3 F(t) \diamond H(t)}\Delta_4,$$

$$\nabla_{24} = \left[\frac{\begin{bmatrix}16\ b_1^3 G^{\diamond 2}(t) - 675 \\ a_{-1}F(t) \diamond H(t)\end{bmatrix}}{100 b_1^2 F(t) \diamond H(t)} + a_{-1}\right],$$

$$\nabla_{25} = \left[\frac{\begin{bmatrix}16\ b_1^3 G^{\diamond 2}(t) - 675 \\ a_{-1}F(t) \diamond H(t)\end{bmatrix}}{100 b_1^2 F(t) \diamond H(t)} - a_{-1}\right]$$

and

$\Delta_4 = [\cos^{\diamond}(2\Xi_2) - i\sin^{\diamond}(2\Xi_2)]$,

$\Delta_5 = [\cos^{\diamond}(\Xi_2) + i\sin^{\diamond}(2\Xi_2)]$,

$\Delta_6 = [\cos^{\diamond}(\Xi_2) - i\sin^{\diamond}(2\Xi_2)]$,

$$U_6(t,x) = \frac{\begin{bmatrix} \nabla_{26} \diamond \cos^{\diamond}(2\Xi_3(t,x)) \\ +i\, \nabla_{27} \diamond \sin^{\diamond}(2\Xi_3(t,x,)) \end{bmatrix}}{\begin{bmatrix} [1+b_{-2}]\cos^{\diamond}(2\Xi_3(t,x)) \\ +i[1-b_{-2}]\sin^{\diamond}(2\Xi_3(t,x)) \end{bmatrix}}, (3.6)$$

$$\nabla_{26} = \left[\frac{\begin{bmatrix} -8kb_{-2}G(t) \\ +a_{-1}F(t) \end{bmatrix}}{b_{-2}F(t)} + a_{-2}\right],$$

$$\nabla_{27} = \left[\frac{\begin{bmatrix} -8kb_{-2}\,G(t) \\ +a_{-1}F(t) \end{bmatrix}}{b_{-2}F(t)} - a_{-2}\right],$$

with

$$\Xi_1(t,x) = k\left[x - \frac{2}{25}\, \nabla_{28}\right], \quad (3.7)$$

$$\nabla_{28} = \int_0^t [\frac{\begin{bmatrix} 50\, a_{-1}F(t) \diamond H(t) \\ -3b_0^2 G^{\diamond 2}(t) \end{bmatrix}}{b_0^2 F(t)}] d\tau$$

$$\Xi_2(t,x) = k\left[x + \frac{3}{100}\, \nabla_{29}\right], \quad (3.8)$$

$$\nabla_{29} = \int_0^t \left[\frac{\begin{bmatrix} 8\, b_1^3 G^{\diamond 2}(t) + 225 \\ a_{-1}F(t) \diamond H(t) \end{bmatrix}}{b_1^3 H(t)}\right] d\tau\,.$$

$$\Xi_3(t,x) = k[x - \nabla_{30}], \qquad (3.9)$$

$$\nabla_{30} = \int_0^t \left[\frac{\begin{bmatrix} -4kb_{-2}\,G(t) + a_{-1} \\ F(t) + 16k^2 b_{-2}H(t) \end{bmatrix}}{b_{-2}}\right] d\tau.$$

We observe that for different forms of $F(t)$, $G(t)$ and $H(t)$, we can get different generalized white noise functional solutions of $Eq.(1.2)$ from $Eqs.(3.1)-(3.6)$.

Remark

It is well known that Wick version of function is usually difficult to evaluate. So, in this section, we give non-Wick version of solutions of $Eq.(1.2)$. Let $W_t = \dot{B}_t$ be the Gaussian white noise, where B_t is the Brownian motion. We have the Hermite transform $\widetilde{W}_t(z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)\,ds$ [24]. Since $\exp^{\diamond}(B_t) = \exp\left(B_t - \frac{t^2}{2}\right)$, we have $\sin^{\diamond}(B_t) = \sin\left(B_t - \frac{t^2}{2}\right)$ and $\cos^{\diamond}(B_t) = \cos\left(B_t - \frac{t^2}{2}\right)$. *Suppose* $F(t) = \mu_1 H(t)$,

$G(t) = \mu_2 H(t)$, and $H(t) = \alpha\ (t) + \mu_3$ Wt + where $\mu_1,\ \mu_2\ and\ \mu_3$ are arbitrary constants and $\alpha(t)$ is integrable or bounded measurable function on R_+.

Therefore, for $F(t)\, G(t)\, H(t) \neq 0$ the generalized white noise functional solutions of $Eq.\,(1.2)$ are as follows:

$$U7(t,x) = \frac{\begin{bmatrix} V_{31} + \frac{4a_{-1}}{b_0} + \\ a_{-1}\exp(-\Psi_1(t,x)) \end{bmatrix}}{\begin{bmatrix} \exp((\Psi_1(t,x)) + b_0 \\ + \frac{b_0^2}{4}\exp(-(\Psi_1(t,x)) \end{bmatrix}}, \qquad (4.1)$$

Where

$$V_{31} = \frac{4}{25}\,\frac{\begin{bmatrix} (25\ a_{-1}\mu_1 \\ -3\mu_2^2\, b_0^2)\boldsymbol{H(t)} \end{bmatrix}}{\mu_1 b_0^2}\exp(\Psi_1(t,x))$$

$$U_8(t,x) = \frac{\begin{vmatrix} V_{32} + V_{33} + a_{-1} \\ \exp(-\Psi_2(t,x,)) \end{vmatrix}}{\begin{bmatrix} \exp(-2\Psi_2(t,x)) + \\ b_1\exp(\Psi_2(t,x)) - \\ \frac{4b_1^3}{27}\exp(-\Psi_2(t,x)) \end{bmatrix}}, (4.2)$$

Where

$$\boldsymbol{V_{32}} = -\left[\frac{\begin{bmatrix} 48\, b_1^3\mu_2^2 + \\ 675 a_{-1}\mu_1 \end{bmatrix}}{100 b_1^3\mu_1}\right]\exp(-2\Psi_2(t,x)),$$

$$\boldsymbol{V_{33}} = \left[\frac{\begin{bmatrix} 16\, b_1^3\mu_2^2 - 675 \\ a_{-1}\mu_1 \end{bmatrix}}{100 b_1^2\mu_1}\right]\exp(\Psi_2(t,x)),$$

$$\boldsymbol{U_9(t,x)} = \frac{|V_{34} + a_{-2}\exp(-2\Psi_3(t,x,))|}{\begin{bmatrix} \exp(2\Psi_3(t,x)) \\ +b_{-2}\exp(-2\Psi_3(t,x,)) \end{bmatrix}}, (4.3)$$

Where

$$\boldsymbol{V_{34}} = \frac{\begin{bmatrix} -8b_{-2}\,\mu_2 \\ +a_{-1}\mu_1 \end{bmatrix}}{b_{-2}\mu_1}\exp(2\Psi_3(t,x)),$$

$$U_{10}(t,x) = \frac{\left[\begin{array}{c} \nabla_{35}\cos((\Psi_1(t,x)) + \\ \frac{4a_{-1}}{b_0} + i\,\nabla_{36}\sin((\Psi_1(t,x)) \end{array}\right]}{\left[\begin{array}{c}\left[1+\frac{b_0^2}{4}\right]\cos((\Psi_1(t,x)) \\ \left[1-\frac{b_0^2}{4}\right] i\sin((\Psi_1(t,x))\end{array}\right]}, (4.4)$$

Where

$$\nabla_{35} = \left[\frac{4}{25}\left[\frac{\left[\begin{array}{c}25\,a_{-1}\mu_1 \\ -3b_0^2\mu_2^2\end{array}\right]H(t)}{b_0^2\mu_1}\right] + a_{-1}\right],$$

$$\nabla_{36} = \left[\frac{4}{25}\left[\frac{\left[\begin{array}{c}25\,a_{-1}\mu_1 \\ -3b_0^2\mu_2^2\end{array}\right]H(t)}{b_0^2\mu_1}\right] - a_{-1}\right].$$

$$U_{11}(t,x) = \frac{\left[\begin{array}{c}\nabla_{37} + \nabla_{38}\cos(\Psi_2(t,x)) \\ +i\,\nabla_{39}\sin(\Psi_2(t,x))\end{array}\right]}{\left[\Delta_7 + b_1\Delta_8 - \frac{4b_1^3}{27}\Delta_9\right]}, \quad (5.5)$$

$$\nabla_{37} = -\frac{\left[\begin{array}{c}48\,b_1^3\mu_2^2 + 675 \\ a_{-1}\mu_1\end{array}\right]}{100b_1^3\mu_1}\,\Delta_7,$$

$$\nabla_{38} = \left[\frac{\left[\begin{array}{c}16\,b_1^3\mu_2^2 - 675 \\ a_{-1}\mu_1\end{array}\right]}{100b_1^2\mu_1} + a_{-1}\right],$$

$$\nabla_{39} = \left[\frac{\left[\begin{array}{c}16\,b_1^3\mu_2^2 - 675 \\ a_{-1}\mu_1\end{array}\right]}{100b_1^2\mu_1} - a_{-1}\right]$$

And

$\Delta_7 = [\cos(2\Psi_2) - i\sin(2\Psi_2)]$,

$\Delta_8 = [\cos(\Psi_2) + i\sin(2\Psi_2)]$,

$\Delta_9 = [\cos(\Psi_2) - i\sin(2\Psi_2)]$,

$$U_{12}(t,x) = \frac{\left[\begin{array}{c}\nabla_{40}\cos(2\Psi_2(t,x)) \\ +i\,\nabla_{41}\sin(2\Psi_2(t,x,))\end{array}\right]}{\left[\begin{array}{c}[1+b_{-2}]\cos(2\Psi_2(t,x)) \\ +i[1-b_{-2}]\sin(2\Psi_2(t,x))\end{array}\right]}, (4.6)$$

$$\nabla_{40} = \left[\frac{\left\lfloor \begin{matrix} -8kb_{-2}\mu_2 \\ +a_{-1}\mu_1 \end{matrix} \right\rfloor}{b_{-2}\mu_1} + a_{-2}\right],$$

$$\nabla_{41} = \left[\frac{\left\lfloor \begin{matrix} -8kb_{-2}\mu_2 \\ +a_{-1}\mu_1 \end{matrix} \right\rfloor}{b_{-2}\mu_1} - a_{-2}\right],$$

with

$$\Psi_1(t,x) = k\left[x - \frac{2}{25}\ \Delta_{10}\nabla_{42}\right], \quad (4.7)$$

$$\Delta_{10} = \frac{[50a_{-1}\mu_1 - 3\mu_2^2 b_0^2]}{b_0^2},$$

$$\nabla_{42} = \int_0^t \alpha(\tau)d\tau + \mu_3\left(B_t - \frac{t^2}{2}\right),$$

$$\Psi_2(t,x) = k\left[x + \frac{3}{100}\ \Delta_{11}\nabla_{42}\right], (4.8)$$

$$\Delta_{11} = \frac{[225a_{-1}\mu_1 + 8\mu_2^2 b_0^3]}{b_0^3}$$

$$\Psi_3(t,x) = k[x - \Delta_{12}\nabla_{42}], \qquad (4.9)$$

$$\Delta_{12} = \frac{[a_{-2}\mu_1 + 16k^2 b_{-2} - 4kb_{-2}\mu_2]}{b_{-2}}$$

Conclusion

This paper is devoted to implement new strategies that give generalized white noise functional solutions for the variable coefficients Wick-type stochastic KdV-Burgers equations. The strategies pursued in this work rest mainly on Hermite transform, white noise analysis and exp-function method, all of which are employed to find generalized white noise functional solutions of $Eq.(1.2)$. Moreover, the planner which we have proposed in this paper can be also applied to other nonlinear $PDEs$ in mathematical physics, e.g., $KdV, mKdV,$ Sawada-Kotera, Zhiber-Shabat, Zakharov-Kuznetsov and BBM equations. In the case of $G(t) = 0, Eq.(1.2)$ is reduced to the Wick-type stochastic KdV equations [14, 31]. So, we can obtain a new set of white noise functional solutions for the Wick-type stochastic KdV equations by setting $G(t) = 0$ in $Eqs.(3.1) - (3.6)$. Note that, there is a unitary mapping between the Gaussian white noise space and the Poisson white noise space, this connection was given by Benth and Gjerde [3]. Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients $F(t), G(t)$ and $H(t)$ are Poisson white noise functions in $Eq.(1.2)$.

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